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REFERENCES

1. KRASOVSKII N.N., Control of Dynamic Systems, Nauka, Moscow, 1985.
2. KRASOVSKII N.N. and TRET'YAKOV V.E., One problem of optimal control minimizing the guaranteed outcome, Izv. Akad. Nauk SSSR, Tekh. Kibern., 2, 1983.
3. KRASOVSKII A.N., Construction of mixed strategies using stochastic programs, PMM, 51, 2, 1987.

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NEW SOLUTIONS OF TWO-DIMENSIONAL STATIONARY EULER EQUATIONS*

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A generalized method of separation of variables is used to obtain new particular solutions for the stream function describing two-dimensional stationary motions of an ideal fluid. Patterns of streamlines are given. The proof of the stability of some of the solutions is based on a theorem due to Arnol'd /1/.

1. In the case of the two-dimensional stationary motion of an ideal fluid, the stream function $\psi(x, y)$ satisfies the equation

$$\psi_{xx} + \psi_{yy} = \omega(\psi) \tag{1.1}$$

where ω is the vorticity. We will seek the solutions of (1.1) using the method of generalized separation of variables:

$$\psi = \alpha(f(x) + g(y)) \tag{1.2}$$

The problem arises here of finding the functions ω, α, admitting of non-trivial separation of variables, i.e. a separation in which neither of the functions f, g is a polynomial of degree two or less.

Substituting expression (1.2) into (1.1), we obtain

$$(f_{xx} + g_{yy})\,\alpha' + (f_x^2 + g_y^2)\alpha'' = \Phi(f + g)$$

where $\Phi = \omega\cdot\alpha$. Since α' is not zero, the last equation can be written in the form

$$X + \beta Y = F(z) \tag{1.3}$$
$$X = f_{xx} + g_{yy},\quad Y = f_x^2 + g_y^2,\quad z = f + g$$
$$\beta = \alpha''/\alpha',\quad F = \Phi/\alpha'$$

We shall call the solution of Eq.(1.3) non-trivial, if the corresponding Eq.(1.1) admits of non-trivial separation of variables.

Differentiating Eq.(1.3) with respect to x and y we obtain a relation which can be written, after dividing it by $f_x g_y$, in the form

$$2\beta'(z)X + \beta''(z)Y = F''(z) \tag{1.4}$$

Eqs.(1.3) and (1.4) can be regarded as a system of linear algebraic equations in the unknowns X, Y. The system is not inconsistent, provided that the equations are either linearly dependent, or uniquely solvable in X, Y. In the latter case we arrive at the relations

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$$Y = G(z), \quad X = H(z) \tag{1.5}$$

where G, H are certain functions. Differentiating the first relation of (1.5) with respect to x and y we find that the mapping G must be linear: $G(z) = az + b$, where $a, b \in R$. This means that the functions f, g, which represent the solutions of Eqs.(1.5) are polynomials of degree not higher than the second. Thus the separation will be non-trivial only provided that Eqs. (1.3) and (1.4) are linearly dependent. The conditions of linear dependence are as follows: $\beta'' = 2\beta\beta'$, $F'' = 2\beta' F$. Integrating the first of these relations yields the equation $\beta' = \beta^2 + c$, where $c \in R$. Consequently we have the following lemma.

Lemma. Eq.(1.3) can have a non-trivial solution only in the case when the functions β, F satisfy the system

$$\beta' = \beta^2 + c, \quad F'' = 2\beta' F \tag{1.6}$$

Let β be the solution of the first equation of (1.6). In this case

$$F(z) = s_1\beta(z) + s_2\beta(z)\int \beta^{-2}(z)dz \tag{1.7}$$

is a general solution of the second equation of (1.6).

Indeed, since $F = \beta$ is a particular solution of the linear second-order equation (the second equation of (1.6)), it follows that the general solution has, according to /2/, the form (1.7).

Note. If the function β is given, then α can be found from the equation

$$\alpha'' = \beta\alpha' \tag{1.8}$$

and the functions Φ, ω can be recovered using the formulas

$$\Phi = \alpha' F, \quad \omega = \Phi\cdot\alpha^{-1} \tag{1.9}$$

Theorem. Eq.(1.1) admits of a non-trivial separation of variables only when its right-hand sides $\omega(\psi)$ have the form

$$a_1\psi \ln\psi + a_2\psi, \quad a_1e^{\psi} + a_2e^{-2\psi}$$
$$a_1 \sin\psi + a_2(\sin\psi \ln(\mathrm{tg}^1/_4\psi)) + 2\sin(^1/_2\psi))$$
$$a_1 \mathrm{sh}\,\psi + a_2(\mathrm{sh}\,\psi \ln(\mathrm{th}(^1/_4\psi)) + 2\,\mathrm{sh}(^1/_2\psi))$$
$$a_1 \mathrm{sh}\,\psi + a_2(\mathrm{sh}\,\psi\, \mathrm{arctg}\, e^{^1/_2\psi} + \mathrm{ch}(^1/_2\psi))$$

where a_1, a_2 are arbitrary constants.

Eq.(1.1) is solved, as usual, to within the accuracy of the equivalence point transformations /3/. The proof is given below, and the corresponding functions α and the first-order differential equations for f, g are also given.

2. Here we will discuss the possible solutions of system (1.6).

Let $\beta = k$, where $k^2 = -c \neq 0$. Then the corresponding function F will be equal to $A_2z + A_1$ where $A_2 = s_2/k$, $A_1 = s_1k$. In this case we obtain from Eq.(1.3), as a result of separation, relations for f and g, which can be reduced to the following first-order equations:

$$f_x^{\,2} = c_1e^{-2kf} + \frac{A_2}{k}f + \frac{A_1 + m}{k} - \frac{A_2}{2k^2}, \quad g_y^{\,2} = c_2e^{-2kg} + \frac{A_2}{k}g - \frac{m}{k} - \frac{A_2}{2k^2}$$

The function α is found from (1.8): $\alpha = c_3e^{kz} + c_4$, where c_3, $c_4 \in R$. Eqs.(1.9) yield the right-hand side of $\omega(\psi)$:

$$\omega(\psi) = A_2(\psi - c_4)\ln[(\psi - c_4)/c_3] + A_1k(\psi - c_4)$$

Using the transport transformation in ψ and the stretching transformation, we reduce the corresponding Eq.(1.1) to the standard form

$$\psi_{xx} + \psi_{yy} = a_1\psi\ln\psi + a_2\psi$$

Separation of variables of the form $\psi(x, y) = \xi(x)\eta(y)$ was used in /4/ to obtain particular solutions of the above equation. Moreover, it was shown of all equations of the type (1.1) only the equations given above can be so separated. Therefore, the above equation will not be discussed here. The case $\beta = 0$ corresponds to a linear equation.

If the constant c in the first equation of (1.6) is zero then $\beta = -(z + c_1)^{-1}$. The relations

$$\alpha = c_3\ln|z + c_1| + c_4, \quad F = A_1(z + c_1)^2 + A_2(z + c_1)^{-1}$$

where $c_3, c_4, A_1, A_2 \in R$, are obtained in the usual manner. The right-hand side is found from relations (1.9):

$$\omega(\psi) = c_2A_1e^{\chi} + c_2A_2e^{-2\chi}, \quad \chi = (\psi - c_3)/c_2$$

After applying the transport and stretching transformations in ψ, the corresponding Eq.(1.1) will have the form

$$\psi_{xx} + \psi_{yy} = a_1e^{\psi} + a_2e^{-2\psi} \tag{2.1}$$

The separation of variables $\psi = \ln(f(x) + g(y))$ for this equation was carried out earlier in /5/. The general solution of Eq.(2.1) was given for the case $a_2 = 0$, in /6/.

Let the constant c in the first equation of (1.6) be negative and equal to $-k^2$ where $k \in R$. Then two cases are possible: $\beta = -k\,\mathrm{th}\,(zk + c_1)$ or $\beta = -k\,\mathrm{cth}\,(zk + c_1)$.

In the first case we have, in accordance with (1.8), $\alpha = c_3 + 2c_2k^{-1}\,\mathrm{arctg}\,e^{kz+c_1}$. Formulas (1.7) and (1.9) yield

$$\Phi = c_2A_1\,\mathrm{sh}\,w/\mathrm{ch}^2w + c_2A_2\,(w\,\mathrm{sh}\,w/\mathrm{ch}^2w - 1/\mathrm{ch}\,w), \quad w = kz + c_1$$

where A_1, A_2 are arbitrary constants.

It remains to invert the function α and determine

$$\omega(\psi) = -{}^1/_2c_2A_1 \sin 2\chi - {}^1/_2c_2A_2\,(\sin 2\chi \ln(\mathrm{tg}\,{}^1/_2\,\chi) + 2\sin\chi)$$
$$\chi = k(\psi - c_3)/c_2$$

The stretching and transport transformations reduce the corresponding Eq.(1.1) to the standard form

$$\psi_{xx} + \psi_{yy} = a_1 \sin\psi + a_2\,(\sin\psi \ln(\mathrm{tg}\,{}^1/_4\psi) + 2\sin{}^1/_2\psi) \tag{2.2}$$

and the function α to the form

$$\alpha = 4\,\mathrm{arctg}\,e^{f+g} \tag{2.3}$$

Now we must produce non-trivial solutions of Eq.(1.3), which, in the present case, we must transform to the form

$$\mathrm{cth}\,(z)(X - a_2) = -a_1 - a_2z + Y \tag{2.4}$$

and differentiate it in x and y. As a result we obtain an expression, which can be written in the form

$$\mathrm{cth}\,(z)(X - a_2) = {}^1/_2\,(f_{xxx}/f_x + g_{yyy}/g_y)$$

Equating the right-hand sides of the last equations, we obtain a relation which can be separated into two equations for f and g:

$$f_{xxx} = 2f_x\,(f_x^2 - a_2f + m - a_1), \quad g_{yyy} = 2g_y\,(g_y^2 - a_2g - m)$$

(m is the separation constant). We seek the solution of the first of these equations in the form $f_x^2 = Q(f)$. As a result we obtain a linear second-order equation with constant coefficients, whose general solution leads to a first-order equation for the function f

$$f_x^2 = \mu_1e^{2f} + \mu_2e^{-2f} + a_2f - m + a_1 \tag{2.5}$$

Similarly we obtain the equation

$$g_y^2 = \eta_1e^{2g} + \eta_2e^{-2g} + a_2g + m \tag{2.6}$$

where η_1, η_2 is a constant, as yet arbitrary. Next we substitute the expressions for f_x^2, g_y^2, and f_{xx}, g_{yy} into relation (2.4). It remains to equate the coefficients accompanying the different degrees of the exponent and establish a connection between the constants $\eta_1 = \mu_2$, $\eta_2 = \mu_1$.

In order to study the case $\beta = -k\,\mathrm{cth}\,(kz + c_1)$, it is sufficient to repeat, with minimum changes, the arguments given above. In place of (2.2) we now obtain

$$\psi_{xx} + \psi_{yy} = a_1\,\mathrm{sh}\,\psi + a_2\,(\mathrm{sh}\,\psi \ln(\mathrm{th}\,{}^1/_4\psi) + 2\,\mathrm{sh}\,{}^1/_2\,\psi) \tag{2.7}$$

which admits of a separation of variables of the type

$$\psi = 2\ln|\,\mathrm{cth}^1/_2\,(f + g)| \tag{2.8}$$

and the functions f, g will be solutions of Eqs.(2.5) and (2.6) with different relations connecting the constants $\eta_1 = -\mu_2$, $\eta_2 = -\mu_1$. Direct substitution can be used to confirm the validity of this assumption.

If the constant c in the first equation of (1.6) is positive, the solution will be $\beta = -\sqrt{c}\,\mathrm{ctg}\,(\sqrt{c}z + c_1)$. Moreover, we shall follow the above scheme and repeat the arguments which yield the standard equation

$$\psi_{xx} + \psi_{yy} = a_1\,\mathrm{sh}\,\psi + 2a_2\,(\mathrm{sh}\,\psi\,\mathrm{arctg}\,l^{1/_2\psi} + \mathrm{ch}\,{}^1/_2\psi) \tag{2.9}$$

We will separate the variables as follows:

$$\psi = 2\ln|\,\mathrm{tg}\,{}^1/_2\,(f + g)| \tag{2.10}$$

Here the functions f, g satisfy equations which differ from (2.5) and (2.6) only in the fact that e^{2h} is replaced by $\sin 2h$, e^{-2h} by $\cos 2h$ $(h = f, g)$, where $\eta_1 = \mu_1$, $\eta_2 = -\mu_2$.

In /7-9/ the equivalent multiplicative separations of variables were used for Eqs.(2.2) and (2.7), but the constant a_2 was equal to zero. It is possible that the separation of the type (2.10) for Eq.(2.8) is new even when $a_2 = 0$. In what follows, the constant a_2 will be assumed to be non-zero everywhere.

3. Constructing the streamline patterns for Eqs.(2.2), (2.7) and (2.9) requires a knowledge of the topology of the level of the corresponding functions $h(x, y) = f(x) + g(y)$. Although solving the differential equations of the type (2.5) can be reduced to inverting the integrals, the qualitative properties of the functions f, g can be studied more simply using the well-known methods of the theory of non-linear oscillations /10/.

The behaviour of the solutions of Eqs.(2.5) and (2.6) is governed by the corresponding potential functions

$$\Pi_f = -\mu_1 e^{2f} - \mu_2 e^{-2f} - a_2 f, \quad \Pi_g = -\eta_1 e^{2g} - \eta_2 e^{-2g} - a_2 g \tag{3.1}$$

If the constants μ_1, μ_2 are negative, then both functions Π_f, Π_g are convex in the downward direction and each has a single minimum point. Therefore the non-constant solutions of Eqs.(2.5) and (2.6) are, in this case, periodic functions.

Fig.1 shows the pattern of stream lines obtained by solving Eqs.(2.5) and (2.6) numiercally for $\mu_1 = \mu_2 = -0.5$, $m = 1.81$, $a_1 = 3.06$, $a_2 = 0.1$. Moreover, we have assumed that $f(0) = g(0) = 0$. The vortex circuits and distributed in a chequered pattern. The period of the function f is 4.31, while that of g is 4.08. This eanbles us to determine the distance between the centres of any two vortices. The maximum value of the velocity vector modulus is 2.14. It can be shown that the qualitative pattern of the stream lines for $\mu_1 < 0$, $\mu_2 < 0$, in the general case, has the form shown in Fig.1, or a pattern obtained by rotating it by 90°. In special cases the boundaries of the neighbouring circuits become common.

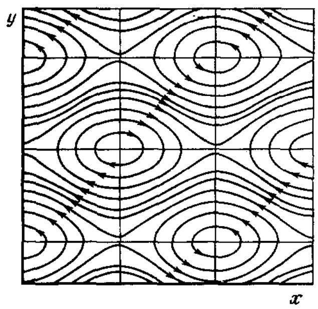

Fig.1

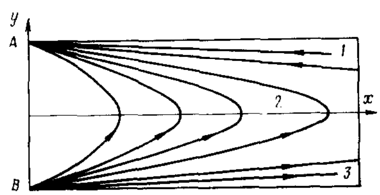

Fig.2

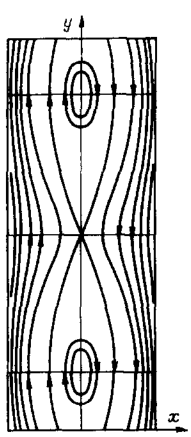

Fig.3

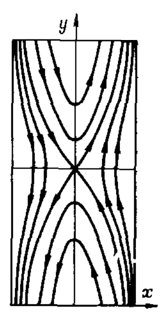

Fig.4

If the constants μ_1, μ_2 are positive, then the functions Π_f, Π_g will be convex in the upward direction and every function will have a unique maximum. This means that we have, in each of the phase planes (f, f_x) and (g, g_y), one saddle-type singularity through which the separatrices pass. The separatrices divide the phase plane into regions filled with twice vanishing trajectories /10/. Since the right-hand sides of Eqs.(2.5) and (2.6) increase exponentially, the twice vanishing trajectories have corresponding solutions with a domain of definition consisting of bounded intervals. The intervals divide into three classes. The first (second) class contains the solutions departing to plus (minus) infinity at both ends of the interval, and the third class consists of solutions tending to minus infinity at one end, and to plus infinity at the other end of the interval. We shall call such solutions the twice vanishing solutions of class 1, 2 or 3. The separatrices have the corresponding solutions whose domain of definition consists of intervals of the form (a, ∞) or $(-\infty, a)$. Their plots have vertical and horizontal asymptotes. The solutions can be of two types, depending on whether the corresponding function tends to plus or minus infinity on approaching the point a. Henceforth, we shall call them the limitingly vanishing solutions of type 1 or 2.

If we take as f and g the limitingly vanishing solutions of type 1, then the corresponding function ψ will determine the motion which can be treated as a flow within a two-sided right angle. When f and g are twice-vanishing solutions of class 1, then the resulting motion can be interpreted as a flow in a rectangular cylinder. In these, as well as in certain other cases, the velocity is bounded everywhere.

The moduli of the velocity vector components are calculated, taking relations (2.3), (2.5) and (2.6) into account, from the formulas $|u| = |\psi_y|, |v| = |\psi_x|$. If the value of g is fixed and $f \to \infty$, then the velocity vector is bounded. When the functions f and g both tend simultaneously to infinity, the velocity tends to zero. The velocity will have a singularity at some point if on approaching this point $f \to \infty$, $g \to -\infty$.

One of the solutions with singularities is shown in Fig.2. The stream lines belong to three regions. Regions *1* and *3* consist of unbounded trajectories emerging from point B, or going into point A. Region *2*, which separates zones *1* and *3*, is filled with trajectories containing points A and B. Although every stream line from region *2* is bounded, arbitrarily long trajectories exist. The pattern of stream lines in question is obtained in the case when the limitingly vanishing solution of type 1 is taken as f, and the twice vanishing solution of class 2 as g. It should be noted that the pattern of stream lines given in /4/ and corresponding to such solutions, is not depicted quite accurately since it does not show the arbitrarily long trajectories lying in region *2*.

Suppose now that the constants μ_1, μ_2 have different signs $\mu_1 > 0$, $\mu_2 < 0$. Then the quantity $a_* = -4\sqrt{-\mu_1\mu_2}$ will become the bifurcation value for Eq.(2.5). Indeed, when $a_2 > a_*$, the first derivative of the function Π_f will be negative. All phase trajectories in the plane (f, f_x) are twice vanishing, and twice vanishing solutions of class 1 correspond to them. If $a_2 = a_*$, then the function Π_f decreases everywhere and at the point of inflection $\Pi_f' = 0$. A higher-order singularity appears in the phase plane and the separatrix passes through it. The limitingly vanishing solutions of type 1 correspond to it. In the case of $a_2 < a_*$ the function Π_f decreases everywhere except at a bounded segment, and has a pair of extrema, i.e. a maximum and a minimum. This means that two singularities exist in the phase plane, a saddle and a centre. Closed trajectories lie near the centre. The separatrix loop enveloping these trajectories emerges from the saddle and returns to it. A smooth bounded solution f corresponds to it, and f tends to a constant value as $x \to \pm\infty$. We shall call such a solution twice limited. Sections of the separatrix also exist which correspond to the limitingly vanishing motions /10/ in the phase plane. The remaining phase trajectories are twice vanishing. We analyse the behaviour of the solutions of (2.6) in the same manner. Here the bifurcation value is $a_2 = -a_*$. When $a_2 < -a_*$, all trajectories in the phase plane are twice vanishing and two singularities, a saddle and a centre, appear when $a_2 > -a_*$.

Qualitative patterns of the stream lines can be obtained by choosing the corresponding functions f, g.

For example, when $a_2 > -a_*$, the function f will be a twice vanishing solution of class 1 and the function g can be taken as periodic.

Fig.3 shows the flow field obtained by solving Eqs.(2.5) and (2.6) numerically for $\mu_1 = -\mu_2 = 0.5$ $a_2 = 3$, $a_1 = m = 0.04$. Moreover, we have assumed that $f(0) = g(0) = 0$. The period of the function g is equal to 4.75. The function f is defined in the interval [-1.32, 1.32]. The solution can be interpreted as a layer of displacement between two parallel walls.

If on the other hand we take the twice limited solution as g, then the pattern of the stream lines will look like the patterns shown in Fig.4. It is clear that here the velocity is a continuous bounded representation.

The case of $\mu_1 < 0$, $\mu_2 > 0$ is analogous to the previous case and will therefore not be discussed. If one of the constants μ_1, μ_2 is equal to zero, then we can show that no new solutions occur.

Since Eq.(2.7) admits of a separation of variables (2.8) with the functions f, g, satisfying the same Eqs.(2.5) and (2.6) (only the relation connecting the constants is different here), it follows that the problem of constructing the level lines of the function $h = f + g$ is similar to that discussed above. New types of flows are possible here, such as, for example, a flow between two sinuous walls periodic in x. The flow appears in the case of $\mu_1 < 0$, $\mu_2 < 0$, when f is a periodic function and g is a twice vanishing solution of class 2. We can describe the whole set of solutions without major difficulties. However, in the course of constructing the patterns of stream lines we must pay special attention to the lines on which the sum $f + g$ is equal to zero, since an approaching these lines the velocity increases without limit.

In conclusion we should mention the solutions of Eq.(2.9) obtained by separating the variables (2.10). The behaviour of the functions f, g is determined here by the potentials

$$\Pi_f = -\mu_1 \sin 2f - \mu_2 \cos 2f - a_2 f, \ \Pi_g = -\mu_1 \sin 2g + \mu_2 \cos 2g - a_2 g$$

This matches qualitatively the behaviour of the function Θ, which is a solution of the equation $\Theta_x^2 = 2\cos\Theta + c_1\Theta + c_2$ analysed in detail in /10/. The bifurcation takes place at $|a_2| = \mu_*$, $\mu_* = 2\sqrt{\mu_1^2 + \mu_2^2}$. If $|a_2| > \mu_*$, then all solutions of the corresponding equations are defined on R, have each a single extremal point, and have an upper or lower limit depending on the sign of a_2. When $|a_2| < \mu_*$, we add to them, such equations as the twice limiting and periodic ones. The case $a_2 = 0$ leads in fact to the equation of a mathematical pendulum. The construction of the corresponding patterns of stream lines is not difficult.

Example. Let the inequality $a_2 > \mu_*$ hold. Then the functions f and g will have no upper limit on R and the minimum points will represent their unique extrema. Therefore the level lines $h = f(x) + g(y)$ will be closed curves, diffeomorphic to the circle. Clearly, a natural number n exists such that the set

$$M_n = \{(x, y) \in R^2: \pi n + \pi/10 \leqslant (f + g)/2 \leqslant \pi n + \pi/3\}$$

will be non-empty and diffeomorphic to the annulus in R^2. If we regard the boundary M_n as solid walls, then a flow between two cylinders will be obtained. When $a_1 > 2a_2$, the derivative in ψ of the right-hand side of $\omega(\psi)$ will satisfy the inequalities $K_1 > \omega' > K_2 > 0$, where K_1, K_2 are suitably chosen constants. All that remains now is to refer to Arnol'ds theorem./1/.

REFERENCES

1. ARNOL'D V.I., On an a priori estimate of the theory of hydrodynamic stability. Izv. Vuz. Matematika, 5 (54), 1966.
2. KAMKE E., Manual of Ordinary Differential Equations. Nauka, Moscow, 1976.
3. OVSYANNIKOV L.V., Group Analysis of Differential Equations. Nauka, Moscow, 1978.
4. SHERCLIFF J.A., Simple rotational flows. J. Fluid Mech., 82, 4, 1977.
5. KAPTSOV O.V., Elliptic solutions of the stationary Euler equations. Dokl. Akad. Nauk SSSR, 298, 3, 1988.
6. VEKUYA I.N., Fundamental of Tensor Analysis and Theory of Covariants. Nauka, Moscow, 1978.
7. KAKPTSOV O.V., Certain classes of plane vortical flows of ideal fluid. PMTF, 1, 1989.
8. TING A.C., CHEN H.H., and LEE Y.C., Exact solutions of non-linear boundary-value problem, the vortices of the two-dimensional sinh-Poisson equation, Physica D. 26, 1/3, 1987.
9. MOVSESYANTS YU.B., Solitons in a two-dimensional hydrodynamic model of cold plasma. Zh. Eksp. Teor. Fiz, 91, 2, 1986.
10. ANDRONOV A.A., VITT A.A. and KHAIKIN S.E., Theory of Oscillations, Nauka, Moscow, 1981.

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